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On the limit behavior of recurrence coefficients for multiple orthogonal polynomials

A.I. Aptekarev[a], V. Kalyagin[b], G. López Lagomasino[c,*], I.A. Rocha[d]

[a] *Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Moscow, Russia*
[b] *Higher School of Economics, Nizhny Novgorod, Russia*
[c] *Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Madrid, Spain*
[d] *Departamento de Matemática Aplicada, E.U.I.T. Telecomunicación, Universidad Politécnica de Madrid, Ctra. de Valencia Km. 7, 28031 Madrid, Spain*

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Dedicated to Barry Simon on the occasion of his sixtieth birthday

Abstract

In this paper we investigate general properties of the coefficients in the recurrence relation satisfied by multiple orthogonal polynomials. The results include as particular cases Angelesco and Nikishin systems.

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1. Introduction

Let $\mu_1, \mu_2, \dots, \mu_m$ be a system of Borel measures with constant sign supported on the real line. For a given multi-index $\mathbf{n} = (n_1, n_2, \dots, n_m) \in \mathbb{Z}_+^m$, the multiple orthogonal polynomial $Q_{\mathbf{n}}(x)$

* Corresponding author.
E-mail addresses: aptekaa@keldysh.ru (A.I. Aptekarev), kalia@hse.nnov.ru (V. Kalyagin), lago@math.uc3m.es (G. López Lagomasino), igalvar@euitt.upm.es (I.A. Rocha).

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is defined by the conditions

$$\deg Q_{\mathbf{n}} \leqslant n := \sum_{k=1}^{m} n_k, \tag{1}$$

$$\int Q_{\mathbf{n}}(x) x^{\nu}\, d\mu_k(x) = 0, \quad \nu = 0, 1, \ldots, n_k - 1, \quad k = 1, \ldots, m. \tag{2}$$

It is well known that $Q_{\mathbf{n}}(x)$ is the common denominator of the Hermite–Padé vector rational approximant

$$\pi_{\mathbf{n}}(z) := \left(\frac{P_{\mathbf{n},1}(z)}{Q_{\mathbf{n}}(z)}, \ldots, \frac{P_{\mathbf{n},m}(z)}{Q_{\mathbf{n}(z)}} \right)$$

for the system of Markov functions $(\widehat{\mu}_1, \ldots, \widehat{\mu}_m)$, where $\widehat{\mu}_k(z) = \displaystyle\int \frac{d\mu_k(x)}{z - x}$, $k = 1, \ldots, m$. This approximant is defined by the conditions

(i) $\deg Q_{\mathbf{n}} \leqslant n = n_1 + \cdots + n_m, \quad Q_{\mathbf{n}} \neq 0,$
(ii) $Q_{\mathbf{n}}(z)\widehat{\mu}_k(z) - P_{\mathbf{n},k}(z) = O\left(\frac{1}{z^{n_k+1}}\right), \quad z \to \infty, \quad k = 1, \ldots, m.$

For $m = 1$ one obtains standard orthogonal polynomials. In the general case, $m > 1$, the multiple orthogonal polynomial $Q_{\mathbf{n}}$ exists for any multi-index but it may not be unique. We say that the multi-index $\mathbf{n} = (n_1, n_2, \ldots, n_m)$ is normal if any polynomial $Q_{\mathbf{n}}$ satisfying (1)–(2) has $\deg Q_{\mathbf{n}} = n$. For a normal multi-index $\mathbf{n}$ the orthogonal polynomial is unique up to a constant factor so the monic polynomial is unique. If, additionally, the zeros of $Q_{\mathbf{n}}$ are real and simple, we say that the multi-index $\mathbf{n}$ is strongly normal.

Multiple orthogonal polynomials are known to be a useful tool, not only in the study of Hermite–Padé rational approximation, but also in analytic number theory, non-symmetric banded difference operators, random matrices, and other directions (see [4,6,15]).

Recurrence relation. Unless otherwise stated, throughout the paper we restrict to the family of multi-indices

$$\mathbf{I} := \{(0, \ldots, 0), (1, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, 1, \ldots, 1), (2, 1, \ldots, 1), \ldots\}.$$

Any multi-index $\mathbf{n}$ in the step-line $\mathbf{I}$ may be identified by the value of $n = n_1 + n_2 + \cdots + n_m$. In fact, if $n = km + s$, $0 \leqslant s \leqslant m - 1$, the corresponding index is $\mathbf{n} = (k+1, \ldots, k+1, k, \ldots, k)$ where the value $k + 1$ is repeated s times. If the multi-indices of $\mathbf{I}$ are all normal, an immediate consequence of the orthogonality conditions (2) is that the associated monic multiple orthogonal polynomials, denoted here simply by Q_n, satisfy the recurrence relation

$$Q_{-m}(x) = \cdots = Q_{-1}(x) = 0, \quad Q_0(x) = 1,$$

$$\begin{aligned} x Q_n(x) = {} & Q_{n+1}(x) + a_{n,n} Q_n(x) + a_{n,n-1} Q_{n-1}(x) + \cdots \\ & + a_{n,n-m} Q_{n-m}(x), \quad n = 0, 1, 2, \ldots \end{aligned} \tag{3}$$

(for more details, see Section 4.1). These relations are a direct generalization for the multiple orthogonal case of the well known three-term recurrence relations satisfied by orthogonal polynomials.

Banded Hessenberg operators. The recurrence relation (3) gives rise to a difference operator on the space $l^2(\mathbb{N})$ defined in the canonical basis by the matrix

$$A = (a_{i,j})_{i,j=0}^{\infty}, \quad a_{i,j} = 0, \ \ j > i+1, \ i > j+m, \quad a_{i,i+1} = 1, \ \ i = 0, 1, \ldots . \tag{4}$$

This is a non-symmetric $(m+2)$ diagonal lower Hessenberg matrix. A very important point is that the spectral properties of this operator are closely connected with the asymptotic properties of the multiple orthogonal polynomials (see [4]). In particular, one can consider the system of measures $\mu_1, \mu_2, \ldots, \mu_m$ generating the relations (3) as the system of spectral measures of the associated Hessenberg operator. This implies the possibility of stating and investigating direct and inverse spectral and scattering problems for this class of operators using advanced results for multiple orthogonal polynomials.

Angelesco and Nikishin systems. In [12], Gonchar, Rakhmanov and Sorokin consider a wide class of multiple orthogonal polynomials defined by systems of measures $(\mu_1, \ldots, \mu_m)$ with the following extreme cases: Nikishin systems in which all the measures have the same support, and Angelesco systems in which the measures are supported on disjoint intervals of the real line $\Delta_1, \ldots, \Delta_m$.

For the formal definition of Angelesco systems no more specifications on the measures are needed. Nikishin systems of measures $(\mu_1, \ldots, \mu_m)$ are defined as follows (we use the notation proposed in [12]).

Let σ_1, σ_2 be two finite Borel measures with constant sign whose supports $\mathrm{supp}(\sigma_1)$, $\mathrm{supp}(\sigma_2)$ are contained in non-intersecting intervals Δ_1, Δ_2, respectively, of the real line $\mathbb{R}$. Set

$$d\langle \sigma_1, \sigma_2 \rangle(x) = \int \frac{d\sigma_2(t)}{x-t}\, d\sigma_1(x).$$

This expression defines a new measure with constant sign whose support coincides with that of σ_1. Let $(\sigma_1, \ldots, \sigma_m)$ be a system of finite Borel measures with constant sign and compact support on the real line. Let Δ_k denote the smallest interval which contains the support of σ_k and assume that $\Delta_k \cap \Delta_{k+1} = \emptyset, \ k = 1, \ldots, m-1$. The system $(\mu_1, \ldots, \mu_m)$ defined by

$$\mu_1 = \sigma_1, \quad \mu_2 = \langle \sigma_1, \sigma_2 \rangle, \quad \ldots, \quad \mu_m = \langle \sigma_1, \langle \sigma_2, \ldots, \sigma_m \rangle\rangle$$

is called the Nikishin system generated by $\sigma_1, \ldots, \sigma_m$ and we write $(\mu_1, \ldots, \mu_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$. The corresponding polynomial $Q_{\mathbf{n}}$ defined for each multi-index $\mathbf{n} = (n_1, \ldots, n_m)$ by the orthogonality conditions (1)–(2) is called the Nikishin polynomial of multiple orthogonality. We wish to say that in the construction of a Nikishin system, consecutive intervals Δ_k may be allowed to have a common end point if the measures σ_k are such that the measures μ_k which they define are finite. Then all the results remain valid. This remark should be taken into consideration in the example at the end of the paper.

In Angelesco's case, any multi-index $\mathbf{n} = (n_1, \ldots, n_m)$ is strongly normal and the corresponding orthogonal polynomial is of degree $n = n_1 + \cdots + n_m$ with n_k simple zeros in the interior of $\Delta_k, \ k = 1, \ldots, m$ (in reference to the interior of intervals we take the Euclidean topology of $\mathbb{R}$). For Nikishin polynomials in [10] it was proved that when $m \leqslant 3$ all multi-indices are strongly normal but for $m > 3$ the problem of the normality for an arbitrary multi-index is still open.

Nevertheless, it is known (see [8,12]) that all multi-indices in

$$\mathbb{Z}_+^m(\circledast) = \{\mathbf{n} \in \mathbb{Z}_+^m : 1 \leqslant i < j \leqslant m \Rightarrow n_j \leqslant n_i + 1\}$$

are strongly normal and the corresponding orthogonal polynomials have all their zeros in the interior of Δ_1. In particular, all indices of $\mathbf{I}$ are normal since $\mathbf{I} \subset \mathbb{Z}_+^m(\circledast)$. Thus the recurrence relation (3) is satisfied for at least Nikishin and Angelesco multiple orthogonal polynomials.

Strong asymptotic of Angelesco and Nikishin orthogonal polynomials were obtained in [1,3], respectively, assuming Szegő's condition on the measures. For Nikishin systems, ratio asymptotic was given in [5] when $\sigma_k' > 0$, a.e. on Δ_k, $k = 1, \ldots, m$. Though both families have ratio asymptotic under rather general assumptions on the measures, the limit functions are not known explicitly and the asymptotic behavior of the coefficients in the recurrence relation (3) is also unknown.

In this paper, we investigate some properties of the recurrence coefficients and of the banded Hessenberg operator defined by them. General conditions on the system of measures are assumed that cover Angelesco and Nikishin systems.

Our first result relates the bound of the recurrence coefficients with some properties of the zeros of the associated multiple orthogonal polynomials.

Theorem 1.1. *Suppose that for a given system of measures* $(\mu_1, \mu_2, \ldots, \mu_m)$, *all indices of* $\mathbf{I}$ *are strongly normal and the zeros of* Q_{n-1} *and* Q_n *interlace for* $n \geqslant n_0$. *Then the recurrence coefficients in* (3) *are uniformly bounded if and only if the zeros of* Q_n, $n \geqslant n_0$, *lie on a bounded interval.*

This result implies the following corollaries.

Corollary 1.1. *The recurrence coefficients of an Angelesco system* $(\mu_1, \ldots, \mu_m)$ *are uniformly bounded if and only if the supports of the measures* μ_k, $k = 1, \ldots, m$, *are compact.*

Corollary 1.2. *The recurrence coefficients of a Nikishin system* $(\mu_1, \ldots, \mu_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ *are uniformly bounded if and only if* σ_1 *has compact support.*

Remark 1.1. In the case of only one measure, Angelesco and Nikishin systems coincide, multiple orthogonal polynomials reduce to orthogonal polynomials, and the associated Hessenberg operator is a tridiagonal Jacobi operator. It is known that the zeros of orthogonal polynomials are real, simple, and the zeros of Q_{n-1} and Q_n interlace for $n \geqslant 1$. In this case, Theorem 1.1 reduces to the well known result that a Jacobi operator is bounded if and only if the zeros of the corresponding orthogonal polynomials lie on a bounded interval (see [7]).

Remark 1.2. For a banded Hessenberg operator the entries $a_{n,n-j}$, $n \geqslant 0$, $j = -1, 0, 1, \ldots, m$, are uniformly bounded if and only if the associated operator is bounded in the Hilbert space $l^2(\mathbb{N})$. Theorem 1.1 and its corollaries may be stated as results of bounding properties of operators. For example, the following assertion is true: the operator associated with an Angelesco system is bounded if and only if the supports of the measures μ_j, $j = 1, 2, \ldots, m$, are compact; and the operator associated with a Nikishin system is bounded if and only if the support of the first measure of the system is compact.

Our second result is related with the limit behavior of the recurrence coefficients for multiple orthogonal polynomials.

Theorem 1.2. *Suppose that for a given system of measures* $(\mu_1, \mu_2, \ldots, \mu_m)$, *all indices in* $\mathbf{I}$ *are strongly normal and the zeros of* Q_{n-1} *and* Q_n *interlace for* $n \geqslant n_0$. *Then, there exist limits with period* $p \in \mathbb{N}$ *of the recurrence coefficients in* (3), *i.e. for each* $i \in \{0, 1, 2, \ldots, p-1\}$ *and* $j \in \{0, 1, 2, \ldots, m\}$ *there exists*

$$\lim_{k\to\infty} a_{pk+i,pk+i-j} = a^0_{i,-j},$$

if and only if there exist analytic functions F_i, $i = 0, 1, \ldots, p-1$, *and a bounded interval* $\Delta \subset \mathbb{R}$, *containing the zeros of all the polynomials* Q_n, *such that*

$$\lim_{k\to\infty} \frac{Q_{kp+i+1}(x)}{Q_{kp+i}(x)} = F_i(x), \quad \mathcal{K} \subset \mathbb{C}\backslash\Delta,$$

uniformly on each compact subset $\mathcal{K}$ *of the indicated region.*

Under the assumptions of Theorem 1.2, from Theorem 1.1, we know that there exists a smallest bounded interval $\Delta \subset \mathbb{R}$ which contains the zeros of all the polynomials Q_{kp+i}. On the other hand, the interlacing property yields that for each $i = 0, \ldots, p-1$, the family of functions $\{Q_{kp+i+1}/Q_{kp+i}\}$, $k \in \mathbb{Z}_+$, is normal on $\mathbb{C}\backslash\Delta$. Therefore, the limit of these ratios takes place uniformly on each compact subset of $\mathbb{C}\backslash\Delta$ if and only if they hold on a neighborhood of ∞.

Corollary 1.3. *Let* $(\mu_1, \ldots, \mu_m) = \mathcal{N}(\sigma_1, \ldots, \sigma_m)$ *and* $\sigma'_k > 0$ *almost everywhere on* Δ_k, $1 \leqslant k \leqslant m$. *Then, for each* $i \in \{0, 1, 2, \ldots, m-1\}$ *and* $j \in \{0, 1, 2, \ldots, m\}$ *there exists*

$$\lim_{k\to\infty} a_{mk+i,mk+i-j} = a^0_{i,-j}.$$

The values of $a^0_{i,-j}$ *only depend on the system of intervals* Δ_k, $k = 1, 2, \ldots, m$.

Notice that in Nikishin systems, in general, the period is $p = m$. The same occurs with Angelesco systems (see [13]). Nevertheless, eventually, other periods may occur.

Remark 1.3. In terms of the associated banded Hessenberg operators, the Corollary above may be stated as follows. Define the entries $a^0_{l,s}$ of a reference Hessenberg operator A^0 by

$$a^0_{k,k+1} = 1, \quad a^0_{mk+i,mk+i-j} = a^0_{i,-j}, \quad k = 0, 1, 2, \ldots, \quad i = 0, 1, 2, \ldots, m-1,$$
$$j = 0, 1, 2, \ldots, m$$

(the rest of the entries equal zero). Then any banded Hessenberg operator associated with a Nikishin system whose measures have the same system of intervals Δ_k, $k = 1, 2, \ldots, m$, and $\sigma'_k > 0$ almost everywhere on Δ_k, is a compact perturbation of the reference operator A^0.

Remark 1.4. In the case $m = 1$ (one measure μ) there is a famous result by Rakhmanov which states that if $\mu' > 0$ almost everywhere on $\mathrm{supp}(\mu) = [-1, 1]$ then the associated Jacobi operator is a compact perturbation of the reference operator A^0 with $a^0_{n,n+1} = 1$, $a^0_{n,n} = 0$, and $a^0_{n,n-1} = 1/4$

(period $m = 1$). For Angelesco systems, the analogue of Corollary 1.3 may be established if the measures defining the Angelesco system satisfy Szegő's condition. In this case, ratio asymptotic is obtained as a consequence of the strong asymptotic proved for Angelesco systems by Aptekarev in [1] (see also [13]). The question of whether or not Angelesco systems have ratio asymptotic when $\mu_k' > 0, k = 1, \ldots, m$, a.e. on Δ_k remains open. Another question of interest in both Angelesco and Nikishin systems is to obtain at least some implicit equations connecting the limit values of the recurrence coefficients and the extreme points of the intervals Δ_k.

The paper is organized as follows. In Section 2, we prove two general Lemmas which imply Theorem 1.1 and its corollaries. Section 3 is devoted to the proof of Theorem 1.2 which is in fact a variation of the direct and inverse Poincaré theorem for recurrence equations. Corollary 1.3 is derived from Theorem 1.2 using a recent result (see [5]) on ratio asymptotic of multiple orthogonal polynomials for Nikishin systems. Finally, in Section 4, we describe some extensions of the previous results and present examples of explicit values for the limits $a_{i,-j}^0$ of Angelesco and Nikishin systems when $m = 2$.

2. Bounding properties of the recurrence coefficients

The proof of Theorem 1.1 is a combination of the following two lemmas.

Lemma 2.1. *Let $\{Q_n\}_{n=0}^{\infty}$ be the sequence of monic polynomials defined by the recurrence relation (possibly with complex coefficients)*

$$xQ_n = Q_{n+1} + a_{n,n}Q_n + a_{n,n-1}Q_{n-1} + \cdots + a_{n,n-m}Q_{n-m}, \quad n = 0, 1, \ldots$$

with initial conditions $Q_0 = 1$, $Q_{-1} = Q_{-2} = \cdots = Q_{-m} = 0$. If the recurrence coefficients $a_{n,n-j}$, $n = 0, 1, 2, \ldots$, $j = -1, 0, 1, 2, \ldots, m$ ($a_{n,n+1} = 1$) are uniformly bounded and

$$M := \sup_{n,j} |a_{n,n-j}| < \infty,$$

then, for $n = 0, 1, \ldots$, the zeros of the polynomials Q_n lie in the disk $|x| \leqslant (m + 2)M$.

Proof. Let us consider the matrix of order $n + 1$

$$A_n := \begin{pmatrix} a_{0,0} & 1 & 0 & \ldots & 0 \\ a_{1,0} & a_{1,1} & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m,0} & a_{m,1} & a_{m,2} & \ldots & 0 \\ 0 & a_{m+1,1} & a_{m+1,2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & a_{n,n} \end{pmatrix}.$$

The eigenvalue problem for the matrix A_n reads as $A_n y = \lambda y$ for some non-zero vector $y = (y_0, y_1, \ldots, y_n)^t$. This gives the following system of equations:

$$\begin{cases} a_{0,0}y_0 + y_1 & = \lambda y_0, \\ a_{1,0}y_0 + a_{1,1}y_1 + y_2 & = \lambda y_1, \\ \cdots\cdots\cdots & \cdots \ \ldots, \\ a_{n-1,n-1-m}y_{n-1-m} + \ldots + a_{n-1,n-1}y_{n-1} + y_n & = \lambda y_{n-1}, \\ a_{n,n-m}y_{n-m} + \ldots\ldots\ldots + a_{n,n}y_n & = \lambda y_n. \end{cases}$$

If we set $y_0 = 0$ the only solution for this system is the trivial one. To find a non-trivial solution, we put $y_0 = 1$ and we get from the first n equations of the system

$$y_1 = Q_1(\lambda), \quad y_2 = Q_2(\lambda), \ldots, y_n = Q_n(\lambda).$$

Thus, the last equation is satisfied if and only if $Q_{n+1}(\lambda) = 0$. This means that the zeros of Q_{n+1} are exactly the eigenvalues of A_n. It is well known that the spectral radius of A_n is bounded above by the operator norm of A_n induced by any vector norm. Taking the sup norm on the vectors, for the zeros $x_{n+1,j}$ of Q_{n+1}, we obtain

$$\max\{|x_{n+1,j}| : j = 1, \ldots, n+1, \ n = 0, 1, \ldots\} \leqslant (m+2)M$$

and the Lemma is proved. $\square$

Lemma 2.2. *Let* $\{Q_n\}_{n=0}^{\infty}$ *be a sequence of monic polynomials such that*:

(i) *for each* $n = 0, 1, \ldots,$ Q_n *has exactly* n *simple zeros which lie on a bounded interval* $\Delta \subset \mathbb{R}$,
(ii) *for each* $n = 0, 1, \ldots$, *between two consecutive zeros of* Q_{n+1} *there is one zero of* Q_n,
(iii) *the polynomials satisfy the* $(m+2)$*-term recurrence relation*

$$xQ_n(x) = Q_{n+1}(x) + a_{n,n}Q_n(x) + a_{n,n-1}Q_{n-1}(x) + \cdots + a_{n,n-m}Q_{n-m}(x),$$
$$n = 0, 1, \ldots .$$

Then, there are positive constants $C(k)$, $k = 0, \ldots, m$, *such that* $|a_{n,n-k}| \leqslant C(k)$ *for all* n.

Proof. Since the zeros interlace, for every compact set $K \subset \mathbb{C} \backslash \Delta$ and each integer j, there is a positive constant $M_j(K)$ such that $|\frac{Q_{n+j}(x)}{Q_n(x)}| \leqslant M_j(K)$, $x \in K$, for all n. Moreover, from the recurrence relation we get

$$x = \frac{Q_{n+1}(x)}{Q_n(x)} + a_{n,n} + a_{n,n-1}\frac{Q_{n-1}(x)}{Q_n(x)} + \cdots + a_{n,n-m}\frac{Q_{n-m}(x)}{Q_n(x)},$$

which means that

$$\lim_{x\to\infty}\left(x - \frac{Q_{n+1}(x)}{Q_n(x)}\right) = a_{n,n}.$$

Take $r > 0$ such that the circle $\gamma_r = [|z| = r|]$ contains the interval Δ in its interior. We obtain

$$a_{n,n} = \frac{1}{2\pi i}\int_{\gamma_r}\left(z - \frac{Q_{n+1}(z)}{Q_n(z)}\right)\frac{dz}{z} = -\frac{1}{2\pi i}\int_{\gamma_r}\frac{Q_{n+1}(z)}{Q_n(z)}\frac{dz}{z}$$

which yields $|a_{n,n}| \leqslant M_1(\gamma_r) = C(0)$.

Let us assume that $|a_{n,n-j}| \leqslant C(j)$, $j = 0, \ldots, k-1$, $k \leqslant m$, and let us prove that the same is true for $|a_{n,n-k}|$. We can write

$$x\,\frac{Q_n(x)}{Q_{n-k}(x)} = \frac{Q_{n+1}(x)}{Q_{n-k}(x)} + \sum_{j=0}^{k-1} a_{n,n-j}\,\frac{Q_{n-j}(x)}{Q_{n-k}(x)} + a_{n,n-k} + \sum_{j=k+1}^{m} a_{n,n-j}\cdot\frac{Q_{n-j}(x)}{Q_{n-k}(x)}.$$

Thus

$$a_{n,n-k} = \lim_{x\to\infty}\left(x\,\frac{Q_n(x)}{Q_{n-k}(x)} - \frac{Q_{n+1}(x)}{Q_{n-k}(x)} - \sum_{j=0}^{k-1} a_{n,n-j}\,\frac{Q_{n-j}(x)}{Q_{n-k}(x)}\right)$$

$$= \frac{1}{2\pi i}\int_{\gamma_r}\left(z\,\frac{Q_n(z)}{Q_{n-k}(z)} - \frac{Q_{n+1}(z)}{Q_{n-k}(z)} - \sum_{j=0}^{k-1} a_{n,n-j}\,\frac{Q_{n-j}(z)}{Q_{n-k}(z)}\right)\frac{dz}{z}$$

which gives the estimate

$$|a_{n,n-k}| \leqslant r M_k(\gamma_r) + M_{k+1}(\gamma_r) + \sum_{j=0}^{k-1} C(j) M_{k-j}(\gamma_r) = C(k)$$

and the proof is complete. □

Proof of Corollary 1.1. It is well known and easy to verify (see, for example, [15]) that in an Angelesco system for any multi-index $\mathbf{n} = (n_1, n_2, \ldots, n_m)$ the multiple orthogonal polynomial $Q_\mathbf{n}$ has exactly n_j simple zeros in the interior of the interval Δ_j and thus $\mathbf{n}$ is strongly normal. We conclude the proof if we show that the interlacing property takes place. This is a consequence of the following lemma which we state in a form appropriate for other applications. Let $\mathbf{n} \in \mathbb{Z}_+^m$. Denote

$$\mathbf{n}^l := (n_1, n_2, \ldots, n_l + 1, \ldots, n_m).$$

Lemma 2.3. *Let $l \in \{1, 2, \ldots, m\}$ and $\mathbf{n}, \mathbf{n}^l$ be strongly normal with respect to the vector measure $(\mu_1, \mu_2, \ldots, \mu_m)$. If for every $a \in \mathbb{R}$, $\mathbf{n}$ is also normal with respect to the vector measure $(v_1, v_2, \ldots, v_m)$, where $dv_j(x) = (x-a)^2\,d\mu_j(x)$, $j = 1, \ldots, m$, then between any two consecutive zeros of $Q_{\mathbf{n}^l}$ there is exactly one zero of $Q_\mathbf{n}$.*

Proof. Consider the polynomial $P = AQ_\mathbf{n} + BQ_{\mathbf{n}^l}$ where A and B are constants such that $|A| + |B| > 0$. Suppose that P has a double zero at $a \in \mathbb{R}$. Then

$$AQ_\mathbf{n} + BQ_{\mathbf{n}^l} = (x-a)^2 R, \quad \deg\, R \leqslant (n-1).$$

The polynomial R satisfies the orthogonality conditions for the multi-index $\mathbf{n}$ with respect to the system of measures $(v_1, \ldots, v_m)$. This is impossible because deg $R \leqslant n-1$ and is not identically zero. Thus, for any constants $A, B, |A| + |B| > 0$, the polynomial P has only simple zeros on the real line. For a fixed value of y, consider the polynomial

$$P_y(x) := Q_{\mathbf{n}^l}(y)Q_{\mathbf{n}}(x) - Q_{\mathbf{n}}(y)Q_{\mathbf{n}^l}(x).$$

This polynomial has a zero at the point $x = y$; consequently, its derivative cannot be equal to zero at the same point. This means that for all $y \in \mathbb{R}$

$$Q_{\mathbf{n}^l}(y)Q'_{\mathbf{n}}(y) - Q_{\mathbf{n}}(y)Q'_{\mathbf{n}^l}(y) \neq 0.$$

Therefore, this expression has constant sign on all $\mathbb{R}$. This implies that $Q_{\mathbf{n}}$ has opposite signs at consecutive zeros of $Q_{\mathbf{n}^l}$. Lemma 2.3 follows and Corollary 1.1 is proved on account of what was said above concerning strong normality of multi-indices for Angelesco systems and the fact that if $(\mu_1, \mu_2, \ldots, \mu_m)$ is an Angelesco system so is $(\nu_1, \nu_2, \ldots, \nu_m)$. □

Proof of Corollary 1.2. Let

$$\mathbf{n} \in \mathbb{Z}_+^m(*) = \{\mathbf{n} \in \mathbb{Z}_+^m : \nexists\, 1 \leqslant i < j < k \leqslant m \text{ with } n_i < n_j < n_k\}.$$

In [9] it was proved that all $\mathbf{n} \in \mathbb{Z}_+^m(*) \supset \mathbf{I}$ are strongly normal, the zeros of the corresponding Nikishin multiple orthogonal polynomials lie in the interior of Δ_1 and satisfy the interlacing property. Thus, Theorem 1.1 applies to Nikishin systems and the corollary follows. □

Remark 2.1. For a Nikishin system $(\mu_1, \ldots, \mu_m) = \mathcal{N}(\sigma_1, \sigma_2, \ldots, \sigma_m)$, the interlacing property may also be derived from Lemma 2.3 since $(\nu_1, \ldots, \nu_m) = \mathcal{N}((x-a)^2 d\sigma_1, d\sigma_2, \ldots, d\sigma_m)$ is also a Nikishin system. For such systems not only the polynomials corresponding to indices in $\mathbb{Z}_+^m(*)$ interlace their zeros, but also the functions of second kind defined inductively by

$$\Psi_{\mathbf{n},0}(z) = Q_{\mathbf{n}}(z), \quad \Psi_{\mathbf{n},k}(z) = \int \frac{\Psi_{\mathbf{n},k-1}(x)}{z-x}\, d\sigma_k(x), \quad k = 1, \ldots, m.$$

For $\mathbf{n} \in \mathbb{Z}_+^m(\circledast) \subset \mathbb{Z}_+^m(*)$ this was proved in [5].

3. Existence of limits for the recurrence coefficients

Proof of Theorem 1.2. The combination of Lemmas 3.1 and 3.2 proves this theorem.

Lemma 3.1. *Let* $\{Q_n\}_{n=0}^{\infty}$ *be a sequence of monic polynomials defined by the recurrence relations* (*possibly with complex coefficients*)

$$xQ_n = Q_{n+1} + a_{n,n}Q_n + a_{n,n-1}Q_{n-1} + \cdots + a_{n,n-m}Q_{n-m}, \quad n = 0, 1, \ldots,$$

with initial conditions $Q_0 = 1$, $Q_{-1} = Q_{-2} = \cdots = Q_{-m} = 0$ *and assume that for each* $i = 0, \ldots, p-1$, *there exist analytic functions* $F_i(x)$ *and a positive constant* R_0 *such that*

$$\lim_{k\to\infty} \frac{Q_{pk+i+1}(x)}{Q_{pk+i}(x)} := F_i(x) = x + F_0^{(i)} + \frac{F_1^{(i)}}{x} + \frac{F_2^{(i)}}{x^2} + \cdots, \quad |x| \geqslant R_0,$$

uniformly on each compact subset of $D = \{x \in \mathbb{C} : |x| > R_0\}$. *Then, there exist limits with period* p *of the recurrence coefficients.*

Proof. For each $i \in \{0, \ldots, p-1\}$, the function $F_i(x)$ is analytic in D with a simple pole at infinity. This implies that, for some $n_0 \in \mathbb{N}$ and $n \geqslant n_0$ the zeros of all the polynomials Q_n lie in the bounded disk $\{x \in \mathbb{C} : |x| \leqslant R_0 + 1\}$. First, we write the power series at infinity for the ratio of the polynomials in the form

$$\frac{Q_{pk+i+1}(x)}{Q_{pk+i}(x)} = x + [F_0^{(i)} + \beta_{0,k}^{(i)}] + \frac{[F_1^{(i)} + \beta_{1,k}^{(i)}]}{x} + \frac{[F_2^{(i)} + \beta_{2,k}^{(i)}]}{x^2} + \cdots, \quad |x| \geqslant R_0.$$

Ratio asymptotic implies that $\beta_{r,k}^{(i)} \to 0$ if $k \to \infty$ for each fixed r and i. Now, we write down the recurrence relation as follows

$$\sum_{j=0}^{m} Q_{pk+i-j}(x) a_{pk+i,pk+i-j} = x Q_{pk+i}(x) - Q_{pk+i+1}(x)$$

and, dividing it by $Q_{pk+i}(x)$, we have

$$\sum_{j=0}^{m} \frac{Q_{pk+i-j}(x)}{Q_{pk+i}(x)} a_{pk+i,pk+i-j} = x - \frac{Q_{pk+i+1}(x)}{Q_{pk+i}(x)}.$$

For $a_{pk+i,pk+i}$ this means

$$a_{pk+i,pk+i} = x - \frac{Q_{pk+i+1}(x)}{Q_{pk+i}(x)} - \sum_{j=1}^{m} \frac{Q_{pk+i-j}(x)}{Q_{pk+i}(x)} a_{pk+i,pk+i-j}.$$

Letting $x \to \infty$, we get

$$a_{pk+i,pk+i} = \lim_{x\to\infty} \left(x - \frac{Q_{pk+i+1}(x)}{Q_{pk+i}(x)} \right) = -[F_0^{(i)} + \beta_{0,k}^{(i)}]$$

which gives the existence of the limit for $a_{pk+i,pk+i}$ and the formulas

$$\lim_{k\to\infty} a_{pk+i,pk+i} = -F_0^{(i)}, \quad i = 0, \ldots, p-1.$$

To obtain the limit of $a_{pk+i,pk+i-1}$ we write the equation

$$\frac{Q_{pk+i-1}(x)}{Q_{pk+i}(x)} a_{pk+i,pk+i-1}$$
$$= \left(x - \frac{Q_{pk+i+1}(x)}{Q_{pk+i}(x)} - a_{pk+i,pk+i} \right) - \sum_{j=2}^{m} \frac{Q_{pk+i-j}(x)}{Q_{pk+i}(x)} a_{pk+i,pk+i-j}.$$

Multiplying by x and making $x \to \infty$ leads to

$$a_{pk+i,pk+i-1} = \lim_{x\to\infty} x \left(x - \frac{Q_{pk+i+1}(x)}{Q_{pk+i}(x)} - a_{pk+i,pk+i} \right) = -[F_1^{(i)} + \beta_{1,k}^{(i)}]$$

and one has the formulas

$$\lim_{k\to\infty} a_{pk+i,pk+i-1} = \lim_{k\to\infty} -[F_1^{(i)} + \beta_{1,k}^{(i)}] = -F_1^{(i)}, \quad i = 0, \ldots, p-1.$$

Analogously, we obtain the following expression for the coefficients $a_{pk+i,pk+i-2}$

$$a_{pk+i,pk+i-2} = \lim_{x\to\infty} x^2 \left(x - \frac{Q_{pk+i+1}(x)}{Q_{pk+i}(x)} - a_{pk+i,pk+i} - \frac{Q_{pk+i-1}(x)}{Q_{pk+i}(x)}\, a_{pk+i,pk+i-1} \right)$$
$$= -[(F_2^{(i)} + \beta_{2,k}^{(i)}) + (F_1^{(i)} + \beta_{1,k}^{(i)})(F_0^{(i-1)} + \beta_{0,k}^{(i-1)})]$$

and we get

$$\lim_{k\to\infty} a_{pk+i,pk+i-2} = \lim_{k\to\infty} -[(F_2^{(i)} + \beta_{2,k}^{(i)}) + (F_1^{(i)} + \beta_{1,k}^{(i)})(F_0^{(i-1)} + \beta_{0,k}^{(i-1)})]$$
$$= -[F_2^{(i)} + F_1^{(i)} F_0^{(i-1)}], \quad i = 0, \ldots, p-1.$$

Repeating the arguments we obtain the existence of the limits as $k \to \infty$ of the recurrence coefficients $a_{pk+i,pk+i-j}$ for $j = 3, 4, \ldots, m$ and the lemma is proved. □

Lemma 3.2. *Let $\{Q_n\}_{n=0}^{\infty}$ be a sequence of monic polynomials defined by the recurrence relations*

$$xQ_n = Q_{n+1} + a_{n,n}Q_n + a_{n,n-1}Q_{n-1} + \cdots + a_{n,n-m}Q_{n-m}, \quad n = 0, 1, \ldots,$$

with initial conditions $Q_0 = 1$, $Q_{-1} = Q_{-2} = \cdots = Q_{-m} = 0$. Suppose that for each $n \in \mathbb{Z}_+$, Q_n has exactly n real simple zeros which interlace those of Q_{n+1}. Assume that for each $i \in \{0, 1, 2, \ldots, p-1\}$ and $j \in \{0, 1, 2, \ldots, m\}$ there exists

$$\lim_{k\to\infty} a_{pk+i,pk+i-j} = a_{i,-j}^0. \tag{5}$$

Then, there exist analytic functions $F_i(x)$, $i = 0, 1, \ldots, p-1$, and an interval $\Delta \subset \mathbb{R}$ such that

$$\lim_{k\to\infty} \frac{Q_{pk+i+1}(x)}{Q_{pk+i}(x)} = F_i(x), \quad \mathcal{K} \subset \mathbb{C}\backslash\Delta,$$

uniformly on each compact subset $\mathcal{K}$ of the indicated region.

Proof. From Lemma 2.1 we know that the zeros of the polynomials Q_n are uniformly bounded and from hypothesis they lie on the real line. Let Δ be the smallest interval containing the zeros of all these polynomials. From the interlacing property, it follows that for each $i \in \{0, 1, 2, \ldots, p-1\}$ the family $\{Q_{pk+i+1}/Q_{pk+i}\}$, $k \in \mathbb{Z}_+$, is normal in $\mathbb{C}\backslash\Delta$. Therefore, to prove the lemma it is sufficient to show that any convergent subsequence has the same limit. This is done, for example, by showing that all limit functions have the same Laurent expansion at infinity which is what we will do.

Fix $i \in \{0, 1, 2, \ldots, p-1\}$. For short, we denote $R_k^{(i)} = Q_{pk+i+1}/Q_{pk+i}$ and

$$R_k^{(i)}(x) = x + R_{0,k}^{(i)} + \frac{R_{1,k}^{(i)}}{x} + \frac{R_{2,k}^{(i)}}{x^2} + \cdots.$$

Since the family of functions $\{R_k^{(i)}\}$, $k \in \mathbb{Z}_+$, is normal in $\mathbb{C}\backslash\Delta$, from Cauchy's formula for the coefficients of a Laurent expansion it follows that for each $r \in \mathbb{Z}_+$ the sequence of coefficients $\{R_{r,k}^{(i)}\}$, $k \in \mathbb{Z}_+$, is uniformly bounded. Using induction on the index r we will prove that for each $r \in \mathbb{Z}_+$ there exists $\lim_{k\to\infty} R_{r,k}^{(i)}$.

Let $r = 0$. We can write the recurrence relation in the form

$$\frac{Q_{pk+i+1}(x)}{Q_{pk+i}(x)} = x - a_{pk+i,pk+i} - \sum_{j=1}^{m} \frac{Q_{pk+i-j}(x)}{Q_{pk+i}(x)} a_{pk+i,pk+i-j}. \tag{6}$$

All the terms in the summation sign have a zero at infinity; therefore, $R_{0,k}^{(i)} = -a_{pk+i,pk+i}$ and

$$\lim_{k\to\infty} R_{0,k}^{(i)} = -a_{i,0}^{0}.$$

Moreover, $i \in \{0, \ldots, p-1\}$ was arbitrary; therefore, the same is true for each i. With this we settle the first step in the induction. Let us assume that for each $i \in \{0, \ldots, p-1\}$, there exists $\lim_{k\to\infty} R_{r,k}^{(i)}$, $r = 0, \ldots, \ell$, $\ell \geqslant 0$. We must show that this is also true for $r = \ell + 1$.

Take $r = \ell+1$, and again we fix $i \in \{0, \ldots, p-1\}$. Let us make some reductions. The sequence of coefficients $\{R_{\ell+1,k}^{(i)}\}$, $k \in \mathbb{Z}_+$, is uniformly bounded. Let Λ be a subsequence of indices such that there exists $\lim_{k\in\Lambda} R_{\ell+1,k}^{(i)}$. The whole sequence of functions $\{Q_{n+1}/Q_n\}$, $n \in \mathbb{Z}_+$, is normal on $\mathbb{C}\backslash\Delta$, passing to a subsequence, if necessary, we may assume without loss of generality that Λ is such that for all $j \in \{0, 1, 2, \ldots, m\}$

$$\lim_{k\in\Lambda} R_k^{(i-j)}(x) = \lim_{k\in\Lambda} \frac{Q_{pk+i-j+1}(x)}{Q_{pk+i-j}(x)} = F_{i-j}(\Lambda; x) = x + F_0^{(i-j)} + \frac{F_1^{(i-j)}}{x} + \frac{F_2^{(i-j)}}{x^2} + \cdots.$$

For the time being, the coefficients on the right hand may depend on Λ. We do not write the dependence to simplify the notation. Nevertheless, from the induction hypothesis and the normality of $\{Q_{n+1}/Q_n\}$, $n \in \mathbb{Z}_+$, we know that

$$\lim_{k\to\infty} R_{r,k}^{(i-j)} = F_r^{(i-j)}, \quad r = 0, \ldots, \ell, \quad j = 0, 1, 2, \ldots, m, \tag{7}$$

which do not depend on Λ. So the first $\ell + 1$ coefficients of each one of these expansions are free from this dependence.

Notice that

$$\frac{Q_{pk+i-j}(x)}{Q_{pk+i}(x)} = \frac{Q_{pk+i-j}(x)}{Q_{pk+i-j+1}(x)} \frac{Q_{pk+i-j+1}(x)}{Q_{pk+i-j+2}(x)} \cdots \frac{Q_{pk+i-1}(x)}{Q_{pk+i}(x)}.$$

Thus,

$$\lim_{k\in\Lambda} \frac{Q_{pk+i-j}(x)}{Q_{pk+i}(x)} = \frac{1}{F_{i-j}(\Lambda; x) F_{i-j+1}(\Lambda; x) \cdots F_{i-1}(\Lambda; x)}, \quad j \geqslant 1,$$

and using (6), we obtain the equation

$$F_i(\Lambda; x) = x - a_{i,0}^0 - \sum_{j=1}^{m} \frac{a_{i,-j}^0}{F_{i-j}(\Lambda; x) F_{i-j+1}(\Lambda; x) \cdots F_{i-1}(\Lambda; x)}. \tag{8}$$

This relation allows us to determine $F_{\ell+1}^{(i)} = \lim_{k\in\Lambda} R_{\ell+1,k}^{(i)}$ and show that the limit is independent of Λ. In fact, $F_{\ell+1}^{(i)}$ is the sum of the coefficients corresponding to $x^{-(\ell+1)}$ of the expansion at infinity of the terms on the right-hand side of this equation.

Consider an expansion at infinity of the form

$$F(x) = x + F_0 + \frac{F_1}{x} + \frac{F_2}{x^2} + \cdots .$$

Let

$$G(x) = \frac{G_1}{x} + \frac{G_2}{x^2} + \cdots \tag{9}$$

be such that $F(x)G(x) \equiv 1$. From the definition of $G(x)$ we have that $G_1 = 1$ and

$$G_{\ell+1} = -(G_1 F_{\ell-1} + G_2 F_{\ell-2} + \cdots + G_\ell F_0), \quad \ell \geqslant 1.$$

This triangular scheme indicates that $G_{\ell+1}$ can be expressed as the sum of products of the coefficients $F_k, k = 0, \ldots, \ell-1$ (since $G_1, \ldots, G_\ell$ can also be expressed in that form). When we multiply two or more functions of the form (9) and expand their product at infinity, each coefficient can be expressed as the sum of products of coefficients corresponding to lower powers of $1/x$ of the functions being multiplied. Using (8), this means that $F^{(i)}_{\ell+1}$ can be expressed in terms of $a^0_{i,-j}$, $j = 0, 1, 2, \ldots, m$, and $F^{(i-j)}_r$, $r = 0, \ldots, \ell-1$, $j = 0, 1, 2, \ldots, m$, which do not depend on Λ (see (5) and (7)). This is true for each $i \in \{0, \ldots, p-1\}$ and every convergent subsequence of indices; therefore,

$$\lim_{k\to\infty} R^{(i)}_{\ell+1,k} = F^{(i)}_{\ell+1}, \quad r = 0, \ldots, \ell, \quad j = 0, 1, 2, \ldots, m,$$

as we needed to prove to conclude the induction and the proof of this lemma. □

Proof of Theorem 1.2. In the particular case when $\{Q_n\}_{n=0}^{\infty}$ is the sequence of monic polynomials of multiple orthogonality associated to a system of measures $(\mu_1, \ldots, \mu_m)$ with respect to the multi-indices $\mathbf{n} \in \mathbf{I}$, Lemma 3.2 gives the reciprocal of the Lemma 3.1 and Theorem 1.2 follows combining these two lemmas. □

Proof of Corollary 1.3. Ratio asymptotic with period m for multiple orthogonal polynomials associated with a Nikishin system such that $\sigma'_k > 0$ almost everywhere in Δ_k, $k = 1, 2, \ldots, m$, was proved in [5]. Therefore, Corollary 1.3 follows from that result and Theorem 1.2. □

Remark 3.1. The proof of Lemma 3.2 gives an algorithm for calculating the coefficients of the functions F_i in terms of the limits (5). From the existence of the limit of the ratios of the orthogonal polynomials it follows that $F_j(x) = F_i(x)$, $j = i \bmod p$, $i = 0, 1, \ldots, p-1$. We can rewrite (8) in the form

$$F_i(x) = x - a^0_{i,0} - \sum_{j=1}^{m} \frac{a^0_{i,-j}}{F_{i-j}(x)F_{i-j+1}(x)\cdots F_{i-1}(x)}$$

and from this equation we can lift the values of the Laurent expansion of the functions $F_i(x)$, $i = 0, 1, \ldots, p-1$, one at a time. They can be expressed as a sum of products of the limits (5). Reciprocally, the proof of Lemma 3.1 shows how to calculate the limits in (5) if there is periodic ratio asymptotic of the sequence of polynomials using their expansion at infinity.

Remark 3.2. The assumptions of Theorem 1.2 imply

$$\lim_{n\to\infty} \frac{Q_{n+p}(x)}{Q_n(x)} = F(x) = \prod_{i=0}^{p-1} F_i(x), \quad \mathcal{K} \subset \mathbb{C}\backslash\Delta,$$

uniformly on each compact subset $\mathcal{K} \subset \mathbb{C}\backslash\Delta$.

4. Some extensions and examples

4.1. Extensions

Let $(\mu_1, \ldots, \mu_m)$ be a system of measures with constant sign on the real line and Δ_k the smallest interval containing $\mathrm{supp}(\mu_k)$. Assume that these intervals are either non-intersecting or coincident and that they are enumerated in such a way that

$$\widetilde{\Delta}_1 = \Delta_1 = \cdots = \Delta_{m_1}, \ldots, \widetilde{\Delta}_j = \Delta_{m-m_j+1} = \cdots = \Delta_m,$$

where $\widetilde{\Delta}_i \cap \widetilde{\Delta}_k = \emptyset$ and $m_1 + \cdots + m_j = m$. Set $m_0 = 0$. Assume that

$$d\mu_{m_0+\cdots+m_{k-1}+i} = g_{k,i}\, d\mu_{m_0+\cdots+m_{k-1}+1}, \quad i = 2, \ldots, m_k, \;\; k = 1, \ldots, j$$

(no relation is required for those k such that $m_k = 1$) where the functions $g_{k,i}$ are continuous on $\widetilde{\Delta}_k$, respectively. Let the functions $g_{k,i}$ be such that for some $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$ and each $k = 1, \ldots, j$, the system of functions

$$\{1, x, \ldots, x^{n_{m_0+m_1+\cdots+m_{k-1}+1}-1}, g_{k,2}, xg_{k,2}, \ldots, x^{n_{m_0+m_1+\cdots+m_{k-1}+2}-1} g_{k,2}, \ldots,$$
$$g_{k,m_k}, xg_{k,m_k}, \ldots, x^{n_{m_0+m_1+\cdots+m_{k-1}+m_k}-1} g_{k,m_k}\}$$

forms a Markov system on $\widetilde{\Delta}_k$. When this is verified, we say that $(\mu_1, \ldots, \mu_m)$ satisfies the Markov property with respect to $\mathbf{n}$, and from the orthogonality relations it is easy to prove that $\mathbf{n}$ is strongly normal and $Q_{\mathbf{n}}$ has exactly $n_{m_1+\cdots+m_{k-1}+1} + \cdots + n_{m_1+\cdots+m_{k-1}+m_k}$ simple zeros in the interior of $\widetilde{\Delta}_k$.

Notice that the Markov property does not depend on the actual measures with constant sign $\mu_1, \mu_{m_1+1}, \ldots, \mu_{m-m_j+1}$, used in the construction of $(\mu_1, \ldots, \mu_m)$. If $(\mu_1, \ldots, \mu_m)$ satisfies the Markov property with respect to $\mathbf{n}$ and $\mathbf{n}^l$, using Lemma 2.3 one obtains that the zeros of $Q_{\mathbf{n}}$ and $Q_{\mathbf{n}^l}$ interlace (automatically $((x-a)^2\, d\mu_1(x), \ldots, (x-a)^2\, d\mu_m(x))$ satisfies the Markov property with respect to $\mathbf{n}$ for each $a \in \mathbb{R}$).

Angelesco and Nikishin systems are limit cases in which the assumptions above may be verified. In an Angelesco system $\Delta_i = \widetilde{\Delta}_i$, $i = 1, \ldots, m$, and the Markov property is trivially guaranteed for any multi-index $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$ since

$$\{1, \ldots, x^{n_i-1}\}, \quad i = 1, \ldots, m,$$

is a Markov system on Δ_i. In a Nikishin system $\Delta_i = \widetilde{\Delta}_1$, $i = 1, \ldots, m$, and the Markov property was proved in [9, Theorem 2] for all $\mathbf{n} \in \mathbb{Z}_+^m(*)$ (for example, if $\mathbf{n} \in \mathbb{Z}_+^m(\circledast)$ one takes $g_{1,i} = \langle \sigma_2, \ldots, \sigma_i \rangle$, $i = 2, \ldots, m$). Intermediate cases arise in the generalized Nikishin systems introduced in [12] which mix Angelesco and Nikishin systems. A simple case of such mixed systems is obtained considering $j\,(> 1)$ Nikishin systems constructed on j non-intersecting intervals of the real line. An example, not related with generalized Nikishin systems, may be built taking $g_{k,i}(x) = e^{\alpha_{k,i}x}, \alpha_{k,i} \neq \alpha_{k,l}, i \neq l, k = 1, \ldots, j$, where $\alpha_{k,i} \in \mathbb{R}$. Here, the Markov property is satisfied with respect to all $\mathbf{n} \in \mathbb{Z}_+^m$. The proof is straightforward fixing k, using induction on m_k, and Rolle's Theorem to reduce the general case to the induction hypothesis by taking derivatives.

We say that $\mathcal{I} \subset \mathbb{Z}_+^m$ is a complete sequence of multi-indices if $|\cdot| : \mathcal{I} \longrightarrow \mathbb{Z}_+$ is a bijection where $|\mathbf{n}| = n_1 + \cdots + n_m, \mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$. On $\mathbb{Z}_+^m$ we establish the following partial order. We say that $\mathbf{n}_1 = (n_1^1, \ldots, n_m^1) < \mathbf{n}_2 = (n_1^2, \ldots, n_m^2)$ if $n_k^1 \leqslant n_k^2, k = 1, \ldots, m$, with strict inequality for at least one component.

Let $\mathcal{I} \subset \mathbb{Z}_+^m$ be a complete totally ordered sequence of multi-indices such that $(\mu_1, \ldots, \mu_m)$ verifies the Markov property for each $\mathbf{n} \in \mathcal{I}$. By $Q_{|\mathbf{n}|}$ we denote the multiple orthogonal polynomial relative to the given system of measures and the multi–index $\mathbf{n} \in \mathcal{I}$. Let $\mathbf{n}_1 = (n_1^1, \ldots, n_m^1) \in \mathcal{I}$ be given. Assume that there exists a $p \in \mathbb{Z}_+$, $|\mathbf{n}_1| - p \geqslant 0$, such that for each $\mathbf{n} = (n_1, \ldots, n_m) \in \mathcal{I}$, $|\mathbf{n}| \leqslant |\mathbf{n}_1| - p - 1$, we have that

$$n_l + 2 \leqslant n_l^1, \tag{10}$$

where $l \in \{1, \ldots, m\}$ is such that $\mathbf{n}^l \in \mathcal{I}$ and $|\mathbf{n}| + 1 = |\mathbf{n}^l|$. If $\mathcal{I} = \mathbf{I}$, it is easy to verify that (10) is satisfied with $p = m$. Nevertheless, there are sequences for which $p \neq m$. For example, the sequence

$$\{(0, 0), (1, 0), (2, 0), (2, 1), (2, 2), (3, 2), (4, 2), (4, 3), \ldots\}$$

satisfies (10) with $p = 3$. Under (10) there exist complex numbers $a_{j,k}$ so that there is the following relation between multiple orthogonal polynomials

$$xQ_{|\mathbf{n}_1|} = Q_{|\mathbf{n}_1|+1} + a_{|\mathbf{n}_1|,|\mathbf{n}_1|}Q_{|\mathbf{n}_1|} + a_{|\mathbf{n}_1|,|\mathbf{n}_1|-1}Q_{|\mathbf{n}_1|-1} + \cdots + a_{|\mathbf{n}_1|,|\mathbf{n}_1|-p}Q_{|\mathbf{n}_1|-p}.$$

In fact, since deg $Q_{|\mathbf{n}|} = |\mathbf{n}|$ for each $\mathbf{n} \in \mathcal{I}$, it follows that

$$xQ_{|\mathbf{n}_1|} = Q_{|\mathbf{n}_1|+1} + a_{|\mathbf{n}_1|,|\mathbf{n}_1|}Q_{|\mathbf{n}_1|} + a_{|\mathbf{n}_1|,|\mathbf{n}_1|-1}Q_{|\mathbf{n}_1|-1} + \cdots + a_{|\mathbf{n}_1|,0}Q_0 \tag{11}$$

($Q_0 \equiv 1$). We must prove that if $|\mathbf{n}_1| \geqslant p + 1$ then $a_{|\mathbf{n}_1|,k} = 0$ if $0 \leqslant k \leqslant |\mathbf{n}_1| - p - 1$. Let $l \in \{1, \ldots, m\}$ be such that $\mathbf{0}^l \in \mathcal{I}$ where $\mathbf{0}$ is the null vector. Integrating both sides of (11) with respect to μ_l we obtain

$$\int xQ_{|\mathbf{n}_1|}(x)\, d\mu_l(x) = a_{|\mathbf{n}_1|,0} \int d\mu_l(x).$$

By assumption $1 \leqslant n_l^1 - 1$, so the integral on the left is equal to zero and so is $a_{|\mathbf{n}_1|,0}$. Suppose we have proved that $a_{|\mathbf{n}_1|,k} = 0$ for all k such that $0 \leqslant k \leqslant N - 1$ and $N \leqslant |\mathbf{n}_1| - p - 1$. Let us show that $a_{|\mathbf{n}_1|,N} = 0$. Let $\mathbf{n} = (n_1, \ldots, n_m) \in \mathcal{I}$ be such that $|\mathbf{n}| = N$. Then, there exists $l \in \{1, \ldots, m\}$ such that $\mathbf{n}^l \in \mathcal{I}$. Eliminate from (11) those terms which by the induction hypothesis we have assumed that their coefficients vanish and integrate both sides of what remains with respect to $x^{n_l}\, d\mu_l(x)$. It follows that

$$\int x^{n_l+1}Q_{|\mathbf{n}_1|}(x)\, d\mu_l(x) = a_{|\mathbf{n}_1|,N} \int x^{n_l}Q_N(x)\, d\mu_l(x). \tag{12}$$

The integral on the right-hand side is different from zero. If this was not true from orthogonality we would obtain that $Q_N \equiv Q_{N+1}$ which is absurd since deg $Q_N <$ deg Q_{N+1}. On the other hand, the integral on the right-hand side equals zero since by assumption $n_l + 1 \leqslant n_l^1 - 1$. Therefore, $a_{|\mathbf{n}_1|,N} = 0$ as needed. What we have just proved has the following immediate consequence.

Proposition 4.1. *Let $\mathcal{I} \subset \mathbb{Z}_+^m$ be a complete totally ordered sequence of multi-indices such that* (10) *takes place for all $\mathbf{n}_1 \in \mathcal{I}$, $|\mathbf{n}_1| - p \geqslant 0$, where $p \in \mathbb{N}$ is fixed. Let $(\mu_1, \dots, \mu_m)$ be a system of measures which satisfies the Markov property with respect to each $\mathbf{n} \in \mathcal{I}$. Then, the corresponding sequence of multiple orthogonal polynomials satisfies a $p+2$ term recurrence relation and consecutive polynomials interlace their zeros; consequently, Theorems* 1.1 *and* 1.2 *hold true.*

If additional to (10) we have that for some $\mathbf{n} \in \mathcal{I}$, $|\mathbf{n}| = |\mathbf{n}_1| - p$, $n_l + 1 = n_l^1$ and $\mathbf{n}_1^l$ is normal with respect to $(\mu_1, \dots, \mu_m)$, then $a_{|\mathbf{n}_1|,|\mathbf{n}_1|-p} \neq 0$ since the integral to the left of (12) must be different from zero.

4.2. Calculation of limits for $m = 2$. Nikishin systems

In this subsection we give some formulas related with the calculation of the limits of the recurrence coefficients in a Nikishin system when $m = 2$. We use the ratio asymptotic of multiple orthogonal polynomials obtained in [5]. Let us recall some results from [5]. Suppose we have a system of two intervals Δ_1, Δ_2 ($\Delta_1 \cap \Delta_2 = \emptyset$) and two measures σ_1, σ_2 with $\mathrm{supp}(\sigma_j) \subset \Delta_j$ and take $(\mu_1, \mu_2) = \mathcal{N}(\sigma_1, \sigma_2)$, where

$$d\mu_1(x) = d\sigma_1(x), \quad d\mu_2(x) = \left(\int_{\Delta_2} \frac{d\sigma_2(t)}{t - x}\right) d\sigma_1(x), \quad x \in \Delta_1.$$

In the sequel we restrict to multi-indices in $\mathbf{I} \subset \mathbb{Z}_+^2$.

Algebraic functions. The ratio asymptotic of multiple orthogonal polynomials Q_n is given in terms of some algebraic functions of order 3. To introduce these functions we consider the 3-sheeted Riemann surface

$$\mathcal{R} = \overline{\bigcup_{k=0}^{2} \mathcal{R}_k},$$

formed by the consecutively "glued'' sheets

$$\mathcal{R}_0 := \overline{\mathbb{C}} \setminus \Delta_1, \quad \mathcal{R}_1 := \overline{\mathbb{C}} \setminus \Delta_1 \cup \Delta_2, \quad \mathcal{R}_2 := \overline{\mathbb{C}} \setminus \Delta_2,$$

where the upper and lower banks of the slits of two neighboring sheets are identified. Let $\psi^{(l)}$, $l = 1, 2$ be a single valued algebraic function on $\mathcal{R}$ whose divisor consists of one simple zero at the point $\infty^{(0)} \in \mathcal{R}_0$ and one simple pole at the point $\infty^{(l)} \in \mathcal{R}_l$. Denote by $\psi_j^{(l)}$, $j = 0, 1, 2$ the branches of the algebraic function $\psi^{(l)}$ corresponding to the different sheets $\mathcal{R}_j$. Suppose we have the following power series expansions of the functions $\psi_j^{(l)}$ at infinity:

$$\psi_0^{(1)}(z) = \frac{c_0^{(1)}}{z} + \frac{c_3^{(1)}}{z^2} + O\left(\frac{1}{z^3}\right), \quad \psi_1^{(1)}(z) = c_1^{(1)} z + c_4^{(1)} + O\left(\frac{1}{z}\right),$$

$$\psi_2^{(1)}(z) = c_2^{(1)} + \frac{c_5^{(1)}}{z} + O\left(\frac{1}{z^2}\right)$$

and

$$\psi_0^{(2)}(z) = \frac{c_0^{(2)}}{z} + \frac{c_3^{(2)}}{z^2} + O\left(\frac{1}{z^3}\right), \quad \psi_1^{(2)}(z) = c_1^{(2)} + \frac{c_4^{(2)}}{z} + O\left(\frac{1}{z^2}\right),$$

$$\psi_2^{(2)}(z) = c_2^{(2)} z + c_5^{(2)} + O\left(\frac{1}{z}\right).$$

This Riemann surface is of genus 0 and any standard symmetric function of the branches $\psi_j^{(l)}(z)$, $j = 1, 2, 3$, is a polynomial on the Riemann sphere $\overline{\mathbb{C}}$. In particular, for $l = 1, 2$ one has

$$\begin{cases} \psi_0^{(l)} + \psi_1^{(l)} + \psi_2^{(l)} = A_l z + B_l, \\ \psi_0^{(l)}\psi_1^{(l)} + \psi_0^{(l)}\psi_2^{(l)} + \psi_1^{(l)}\psi_2^{(l)} = C_l z + D_l, \\ \psi_0^{(l)}\psi_1^{(l)}\psi_2^{(l)} = E_l, \end{cases}$$

where

$$A_1 = c_1^{(1)}, \quad B_1 = c_2^{(1)} + c_4^{(1)}, \quad C_1 = c_1^{(1)} c_2^{(1)},$$

$$D_1 = c_0^{(1)} c_1^{(1)} + c_1^{(1)} c_5^{(1)} + c_2^{(1)} c_4^{(1)}, \quad E_1 = c_0^{(1)} c_1^{(1)} c_2^{(1)}$$

and

$$A_2 = c_2^{(2)}, \quad B_2 = c_1^{(2)} + c_5^{(2)}, \quad C_2 = c_1^{(2)} c_2^{(2)},$$

$$D_2 = c_0^{(2)} c_2^{(2)} + c_1^{(2)} c_5^{(2)} + c_2^{(2)} c_4^{(2)}, \quad E_2 = c_0^{(2)} c_1^{(2)} c_2^{(2)}.$$

Then, the functions $\psi^{(l)}(z)$ satisfy the following algebraic equations:

$$\lambda^3 - (A_l z + B_l)\lambda^2 + (C_l z + D_l)\lambda - E_l = 0, \quad l = 1, 2.$$

Notice that it is easy to find the constants $c_j^{(l)}$, $j = 0, 1, \ldots, 5$ from the coefficients A_l, B_l, C_l, D_l, E_l. In fact

$$c_0^{(1)} = \frac{E_1}{C_1}, \quad c_1^{(1)} = A_1, \quad c_2^{(1)} = \frac{C_1}{A_1}, \quad c_3^{(1)} = -\frac{E_1}{C_1^2}\left[D_1 - E_1 \frac{A_1}{C_1}\right],$$

$$c_4^{(1)} = B_1 - \frac{C_1}{A_1}$$

and

$$c_0^{(2)} = \frac{E_2}{C_2}, \quad c_1^{(2)} = \frac{C_2}{A_2}, \quad c_2^{(1)} = A_2, \quad c_3^{(2)} = -\frac{E_2}{C_2^2}\left[D_2 - E_2 \frac{A_2}{C_2}\right],$$

$$c_5^{(2)} = B_2 - \frac{C_2}{A_2}.$$

Ratio asymptotic. For a given function F which has a Laurent expansion at infinity of the form $F(z) = Cz^k + O(z^{k-1})$, $C \neq 0$, $k \in \mathbb{Z}$, we define $\tilde{F} := F/C$. One of the main results of [5] applied to the case $m = 2$ with $\sigma_j' > 0$ a.e. on Δ_j, $j = 1, 2$, states that the following ratio asymptotic holds uniformly on a compact subsets of $C \setminus \Delta_1$:

$$\lim_{k\to\infty} \frac{Q_{2k+1}}{Q_{2k}} = \tilde{\psi}_1^{(1)} \tilde{\psi}_2^{(1)} = \frac{1}{\tilde{\psi}_0^{(1)}}, \quad \lim_{k\to\infty} \frac{Q_{2k}}{Q_{2k-1}} = \tilde{\psi}_1^{(2)} \tilde{\psi}_2^{(2)} = \frac{1}{\tilde{\psi}_0^{(2)}}$$

and

$$\lim_{k\to\infty} \frac{Q_{2k+1}}{Q_{2k-1}} = \frac{1}{\tilde{\psi}_0^{(1)} \tilde{\psi}_0^{(2)}}, \quad \lim_{k\to\infty} \frac{Q_{2k+2}}{Q_{2k}} = \frac{1}{\tilde{\psi}_0^{(1)} \tilde{\psi}_0^{(2)}}.$$

By Theorem (1.2), we conclude that

$$\lim_{k\to\infty} a_{2k+i,2k+i-j} = a_{i,-j}^N, \quad i = 0, 1, \quad j = 0, 1, 2,$$

where the upper index N stands for Nikishin system.

Calculation of the limits. For a given n we write down the recurrence equation at three different points x_0, x_1, x_2 of $\mathcal{C} \setminus \Delta_1$

$$a_{n,n} Q_n(x_s) + a_{n,n-1} Q_{n-1}(x_s) + a_{n,n-2} Q_{n-2}(x_s) = x_s Q_n(x_s) - Q_{n+1}(x_s),$$
$$s = 0, 1, 2.$$

Then we divide the equations by $Q_n(x_s)$ and make $n \to \infty$ in two different ways: $n = 2k$, $n = 2k + 1$. In the limit, we obtain two systems of equations, one system for the limit values $a_{0,0}^N$, $a_{0,-1}^N$, $a_{0,-2}^N$ and another one for the limit values $a_{1,0}^N$, $a_{1,-1}^N$, $a_{1,-2}^N$:

$$a_{0,0}^N + a_{0,-1}^N \tilde{\psi}_0^{(2)}(x_s) + a_{0,-2}^N (\tilde{\psi}_0^{(1)} \tilde{\psi}_0^{(2)})(x_s) = x_s - (\tilde{\psi}_1^{(1)} \tilde{\psi}_2^{(1)})(x_s), \quad s = 0, 1, 2$$

and

$$a_{1,0}^N + a_{1,-1}^N \tilde{\psi}_0^{(1)}(x_s) + a_{0,-2}^N (\tilde{\psi}_0^{(1)} \tilde{\psi}_0^{(2)})(x_s) = x_s - (\tilde{\psi}_1^{(2)} \tilde{\psi}_2^{(2)})(x_s), \quad s = 0, 1, 2.$$

By Crammer's rule, one can write the solution of each system of equations as the ratio of some determinants of order 3. Now, we make $x_s \to \infty$ in the following ways: $x_0 \to \infty_0 \in \mathcal{R}_0$, $x_1 \to \infty_1 \in \mathcal{R}_1$, $x_2 \to \infty_2 \in \mathcal{R}_2$. The value of the determinants when $x_s \to \infty_s$, $s = 0, 1, 2$, gives the following.

Lemma 4.1. *Let $\sigma_j' > 0$ a.e. on Δ_j, $j = 1, 2$. We have*

$$a_{0,0}^N = \frac{c_3^{(1)}}{c_0^{(1)}}, \quad a_{0,-1}^N = \left[\frac{c_0^{(2)}}{c_2^{(2)}} - \frac{c_2^{(1)} c_0^{(2)}}{c_1^{(1)} c_1^{(2)}} \right], \quad a_{0,-2}^N = \frac{c_0^{(1)} c_0^{(2)}}{c_1^{(1)} c_1^{(2)}},$$

$$a_{1,0}^N = \frac{c_3^{(2)}}{c_0^{(2)}}, \quad a_{1,-1}^N = \left[\frac{c_0^{(1)}}{c_1^{(1)}} - \frac{c_0^{(1)} c_1^{(2)}}{c_2^{(1)} c_2^{(2)}} \right], \quad a_{1,-2}^N = \frac{c_0^{(1)} c_0^{(2)}}{c_2^{(1)} c_2^{(2)}}.$$

These formulas imply

Corollary 4.1. *Let $\sigma_j' > 0$ a.e. on Δ_j, $j = 1, 2$. In terms of the coefficients of the algebraic equations satisfied by the functions $\psi^{(l)}$, $l = 1, 2$ one has*

$$a_{0,0}^N = -\frac{1}{C_1}\left[D_1 - E_1\frac{A_1}{C_1}\right], \quad a_{0,-1}^N = \left[\frac{E_2}{A_2C_2} - \frac{E_2A_2C_1}{A_1^2C_2^2}\right], \quad a_{0,-2}^N = \frac{E_1A_2E_2}{A_1C_1C_2^2},$$

$$a_{1,0}^N = -\frac{1}{C_2}\left[D_2 - E_2\frac{A_2}{C_2}\right], \quad a_{1,-1}^N = \left[\frac{E_1}{A_1C_1} - \frac{E_1A_1C_2}{A_2^2C_1^2}\right], \quad a_{1,-2}^N = \frac{E_1A_1E_2}{A_2C_2C_1^2}.$$

4.3. Calculation of limits for $m = 2$. Angelesco systems

This is the case, when the supports of the measures μ_1 and μ_2 are disjoint intervals $I_1 = [u_1, v_1]$, $I_2 = [u_2, v_2]$, $u_1 < v_1 \leqslant u_2 < v_2$. When $\mu_j \in \mathbf{Reg}$ on I_j, $j = 1, 2$ (for the definition of the class **Reg** of measures see [16]), the weak asymptotic of multiple orthogonal polynomials [11] is determined by an equilibrium problem for a vector potential in the presence of an external field. The supports of the measures which solve this equilibrium problem may not coincide with the intervals I_1, I_2 (see [2] for more details). Denote these supports by Δ_1, Δ_2.

Algebraic functions. It is known that strong asymptotic (see [3]) and ratio asymptotic (see [13]) in this case is related with the same Riemann surface $\mathcal{R}$ as the one defined above for a Nikishin system associated with Δ_1, Δ_2, but the algebraic functions are different. Let $\phi^{(0)}$ be a single valued algebraic function on $\mathcal{R}$ whose divisor consists of one simple pole at the point $\infty^{(0)} \in \mathcal{R}_0$ and one simple zero at the point $\infty^{(1)} \in \mathcal{R}_1$ and $\phi^{(2)}$ be a single valued algebraic function on $\mathcal{R}$ whose divisor consists of one simple pole at the point $\infty^{(2)} \in \mathcal{R}_2$ and one simple zero at the point $\infty^{(1)} \in \mathcal{R}_1$. Denote by $\phi_j^{(0)}$, $j = 0, 1, 2$, and $\phi_j^{(2)}$, $j = 0, 1, 2$, the branches of the algebraic functions $\phi^{(0)}$, $\phi^{(2)}$ corresponding to the different sheets $\mathcal{R}_j$. Suppose we have the following power series expansions of the functions $\phi_j^{(l)}$ at infinity:

$$\phi_0^{(0)}(z) = s_0^{(0)}z + s_3^{(0)} + O\left(\frac{1}{z}\right), \quad \phi_1^{(0)}(z) = \frac{s_1^{(0)}}{z} + \frac{s_4^{(0)}}{z^2} + O\left(\frac{1}{z^3}\right),$$

$$\phi_2^{(0)}(z) = s_2^{(0)} + \frac{s_5^{(0)}}{z} + O\left(\frac{1}{z^2}\right)$$

and

$$\phi_0^{(2)}(z) = s_0^{(2)} + \frac{s_3^{(2)}}{z} + O\left(\frac{1}{z^2}\right), \quad \phi_1^{(2)}(z) = \frac{s_1^{(2)}}{z} + \frac{s_4^{(2)}}{z^2} + O\left(\frac{1}{z^3}\right),$$

$$\phi_2^{(2)}(z) = s_2^{(2)}z + s_5^{(2)} + O\left(\frac{1}{z}\right).$$

The Riemann surface is of genus 0 and any standard symmetric function of the branches $\phi_j^{(l)}(z)$, $j = 0, 1, 2$, is a polynomial on the Riemann sphere $\overline{\mathbb{C}}$. In particular one has

$$\begin{cases} \phi_0^{(0)} + \phi_1^{(0)} + \phi_2^{(0)} = A_0z + B_0, \\ \phi_0^{(0)}\phi_1^{(0)} + \phi_0^{(0)}\phi_2^{(0)} + \phi_1^{(0)}\phi_2^{(0)} = C_0z + D_0, \\ \phi_0^{(0)}\phi_1^{(0)}\phi_2^{(0)} = E_0, \end{cases}$$

$$\begin{cases} \phi_0^{(2)} + \phi_1^{(2)} + \phi_2^{(2)} = A_3 z + B_3, \\ \phi_0^{(2)}\phi_1^{(2)} + \phi_0^{(2)}\phi_2^{(2)} + \phi_1^{(2)}\phi_2^{(2)} = C_3 z + D_3, \\ \phi_0^{(2)}\phi_1^{(2)}\phi_2^{(2)} = E_3, \end{cases}$$

where

$$A_0 = s_0^{(0)}, \quad B_0 = s_3^{(3)} + s_2^{(0)}, \quad C_0 = s_0^{(0)} s_2^{(0)},$$

$$D_0 = s_0^{(0)} c_5^{(0)} + s_3^{(0)} s_2^{(0)} + s_0^{(0)} s_1^{(0)}, \quad E_0 = s_0^{(0)} s_1^{(0)} s_2^{(0)},$$

and

$$A_3 = s_2^{(2)}, \quad B_3 = s_0^{(2)} + s_5^{(2)}, \quad C_3 = s_0^{(2)} s_2^{(2)},$$

$$D_3 = s_0^{(2)} s_5^{(2)} + s_3^{(2)} s_2^{(2)} + s_1^{(2)} s_2^{(2)}, \quad E_3 = s_0^{(2)} s_1^{(2)} s_2^{(2)}.$$

Thus, the functions $\phi^{(0)}(z)$ and $\phi^{(2)}(z)$ satisfy the following algebraic equations:

$$\lambda^3 - (A_0 z + B_0)\lambda^2 + (C_0 z + D_0)\lambda - E_0 = 0,$$

$$\lambda^3 - (A_3 z + B_3)\lambda^2 + (C_3 z + D_3)\lambda - E_3 = 0.$$

It is easy to find the constants $s_j^{(l)}$, $j = 0, 1, \ldots, 5$, from the coefficients A_l, B_l, C_l, D_l, E_l. In fact,

$$s_0^{(0)} = A_0, \quad s_1^{(0)} = \frac{E_0}{C_0}, \quad s_2^{(0)} = \frac{C_0}{A_0}, \quad s_3^{(0)} = B_0 - \frac{C_0}{A_0},$$

$$s_4^{(0)} = -\frac{E_0}{C_0^2}\left[D_0 - E_0 \frac{A_0}{C_0}\right],$$

and

$$s_0^{(2)} = \frac{C_3}{A_3}, \quad s_1^{(2)} = \frac{E_3}{C_3}, \quad s_2^{(2)} = A_3, \quad s_4^{(2)} = -\frac{E_3}{C_3^2}\left[D_3 - E_3 \frac{A_3}{C_3}\right],$$

$$s_5^{(2)} = B_3 - \frac{C_3}{A_3}.$$

Ratio asymptotic. For a given function F which has a Laurent expansion at infinity of the form $F(z) = Cz^k + O(z^{k-1})$, $C \neq 0$, $k \in \mathbb{Z}$, we define as above $\tilde{F} := F/C$. If the measures μ_1, μ_2 satisfy Szegő's condition on I_1, I_2 it is known (see [2,13]) that we have ratio asymptotic for the multiple orthogonal polynomials Q_n, uniformly on a compact subsets of $\overline{\mathbb{C}} \setminus \Delta_1 \cup \Delta_2$

$$\lim_{k\to\infty} \frac{Q_{2k}}{Q_{2k+1}} = \tilde{\phi}_1^{(0)}, \quad \lim_{k\to\infty} \frac{Q_{2k-1}}{Q_{2k}} = \tilde{\phi}_1^{(2)},$$

and

$$\lim_{k\to\infty} \frac{Q_{2k-1}}{Q_{2k+1}} = \tilde{\phi}_1^{(0)} \tilde{\phi}_1^{(2)}, \quad \lim_{k\to\infty} \frac{Q_{2k}}{Q_{2k+2}} = \tilde{\phi}_1^{(0)} \tilde{\phi}_1^{(2)}.$$

By Theorem (1.2), we conclude that

$$\lim_{k\to\infty} a_{2k+i,2k+i-j} = a^A_{i,-j}, \quad i = 0, 1, \quad j = 0, 1, 2,$$

where the upper index A stands for Angelesco system.

Calculation of limits. For a given n, we write down the recurrence equation at three different points x_0, x_1, x_2 from $\mathbb{C}\backslash\Delta_1 \cup \Delta_2$

$$a_{n,n}Q_n(x_s) + a_{n,n-1}Q_{n-1}(x_s) + a_{n,n-2}Q_{n-2}(x_s) = x_s Q_n(x_s) - Q_{n+1}(x_s),$$
$$s = 0, 1, 2.$$

Divide the equations by $Q_n(x_s)$ and let $n \to \infty$ in two different ways: $n = 2k$, $n = 2k + 1$. At the limit, we obtain two systems of equations, one system for the limit values $a^A_{0,0}, a^A_{0,-1}, a^A_{0,-2}$ and another one for the limit values $a^A_{1,0}, a^A_{1,-1}, a^A_{1,-2}$. Namely,

$$a^A_{0,0} + a^A_{0,-1}\tilde{\phi}_1^{(2)}(x_s) + a^A_{0,-2}\tilde{\phi}_1^{(0)}\tilde{\phi}_1^{(2)}(x_s) = x_s - \frac{1}{\tilde{\phi}_1^{(0)}(x_s)}, \quad s = 0, 1, 2,$$

and

$$a^A_{1,0} + a^A_{1,-1}\tilde{\phi}_1^{(0)}(x_s) + a^A_{0,-2}\tilde{\phi}_1^{(0)}\tilde{\phi}_1^{(2)}(x_s) = x_s - \frac{1}{\tilde{\phi}_1^{(2)}(x_s)}, \quad s = 0, 1, 2.$$

By Crammer's rule, one can write the solutions of each system of equations as the ratio of some determinants of order 3. Now, let $x_s \to \infty$ in the following ways: $x_0 \to \infty_0 \in \mathcal{R}_0$, $x_1 \to \infty_1 \in \mathcal{R}_1$, $x_2 \to \infty_2 \in \mathcal{R}_2$. The limit values of the determinants as $x_s \to \infty_s$, $s = 0, 1, 2$, give

Lemma 4.2. *Let μ_1, μ_2 satisfy Szegő's condition on I_1, I_2, respectively. Then*

$$a^A_{0,0} = \frac{s_4^{(0)}}{s_1^{(0)}}, \quad a^A_{0,-1} = \left[\frac{s_1^{(2)}}{s_2^{(2)}} - \frac{s_2^{(0)}s_1^{(2)}}{s_0^{(0)}s_0^{(2)}}\right], \quad a^A_{0,-2} = \frac{s_1^{(0)}s_1^{(2)}}{s_0^{(0)}s_0^{(2)}},$$

$$a^A_{1,0} = \frac{s_4^{(2)}}{s_1^{(2)}}, \quad a^A_{1,-1} = \left[\frac{s_1^{(0)}}{s_0^{(0)}} - \frac{s_1^{(0)}s_0^{(2)}}{s_2^{(0)}s_2^{(2)}}\right], \quad a^A_{1,-2} = \frac{s_1^{(0)}s_1^{(2)}}{s_2^{(0)}s_2^{(2)}}.$$

From these formulas one obtains

Corollary 4.2. *Let μ_1, μ_2 satisfy Szegő's condition on I_1, I_2, respectively. Then, in terms of the coefficients of the algebraic equations satisfied by the functions $\phi^{(l)}$, $l = 0, 2$ one has*

$$a^A_{0,0} = -\frac{1}{C_0}\left[D_0 - E_0\frac{A_0}{C_0}\right], \quad a^A_{0,-1} = \left[\frac{E_3}{A_3C_3} - \frac{E_3A_3C_0}{A_0^2C_3^2}\right], \quad a^A_{0,-2} = \frac{E_0A_3E_3}{A_0C_0C_3^2},$$

$$a^A_{1,0} = -\frac{1}{C_3}\left[D_3 - E_3\frac{A_3}{C_3}\right], \quad a^A_{1,-1} = \left[\frac{E_0}{A_0C_0} - \frac{E_0A_0C_3}{A_3^2C_0^2}\right], \quad a^A_{1,-2} = \frac{E_0A_0E_3}{A_3C_3C_0^2}.$$

4.4. Connection between Angelesco and Nikishin systems for $m = 2$

There exists a clear connection between the functions $\phi^{(0)}$, $\phi^{(2)}$ and $\psi^{(1)}$, $\psi^{(2)}$. Indeed, the function $1/\phi^{(0)}$ is holomorphic on $\mathcal{R}$, has only one zero at ∞_0 and only one pole at ∞_1; that is, the same divisor as the function $\psi^{(1)}$. The function $\phi^{(2)} - \phi_0^{(2)}(\infty)$ has only one zero at ∞_0 and only one pole at ∞_2; that is, the same divisor as the function $\psi^{(1)}$. This implies the relations

$$\psi^{(1)} = C_1 \frac{1}{\phi^{(0)}}, \quad \psi^{(2)} = C_2(\phi^{(2)} - \phi_0^{(2)}(\infty)),$$

where C_1, C_2 are appropriate constants. Therefore, if we know the power series expansions for one case (Angelesco or Nikishin) we can find the power series expansions and limits of the recurrence coefficients for the other case. This idea is applied in the next subsection to get the explicit values of the limits in one interesting situation.

4.5. A particular case for $m = 2$

There is a particular type of Angelesco system where all can be calculated explicitly. This allows us to find the limits of the recurrence coefficients for the associated Nikishin systems. Consider two intervals $I_1 = [a, 0]$, $I_2 = [0, 1]$ with $-1 \leqslant a < 0$ and two measures μ_1, μ_2 satisfying Szegő's conditions on I_1, I_2. It is shown in [13] that the recurrence coefficients in this case are limit periodic with the period 2. More precisely, in [14] the following formulas were obtained (the upper index A stands for Angelesco system):

$$a_{0,0}^A = \frac{a+1}{3} - \frac{2}{9}R, \quad a_{1,0}^A = \frac{a+1}{3} + \frac{2}{9}R,$$

$$a_{0,-1}^A = \frac{4}{81}R^2, \quad a_{1,-1}^A = \frac{4}{81}R^2,$$

$$a_{0,-2}^A = \frac{4}{729}(K + 2R^3), \quad a_{1,-2}^A = \frac{4}{729}(K - 2R^3),$$

where $R := \sqrt{a^2 - a + 1}$, and $K := 2a^3 - 3a^2 - 3a + 2$.

The Riemann surface $\mathcal{R}$ in this case is associated with the intervals $\Delta_1 = [a, 0]$, $\Delta_2 = [x_a, 1]$, where $x_a = (a+1)^3/9(a^2 - a + 1)$. The function $w := 1/(\phi^{(0)}\phi^{(2)})$ is holomorphic on the Riemann surface $\mathcal{R}$ and satisfies the algebraic equation

$$w^3 - P_1(x)w^2 + P_2(x)w - P_3 = 0, \tag{13}$$

where

$$P_1(x) := [x^2 - (2/3)(1+a)x - (1/27)(a^2 - 10a + 1)],$$

$$P_2(x) := -(2/9)^3[(a^3 - 4a^2 + a) + (-2 + 3a + 3a^2 - 2a^3)x],$$

$$P_3(z) := -2(2/27)^3(a^2 - 2a^3 + a^4).$$

The function $w(x)$ has its branch points at a, 0, 1, and at the point x_a. All these branch points are of second order. One branch of $w(x)$ has a pole of order 2 at infinity and the two other branches have there simple zeros. If we denote the branches by $w_0(x)$, $w_1(x)$, and $w_2(x)$, then $w_1(x)$ is meromorphic in the extended complex plane $\overline{\mathbb{C}}$ with cuts over $[a, 0]$ and $[x_a, 1]$, $w_0(x)$ is holomorphic in $\overline{\mathbb{C}}$ with a cut over $[a, 0]$ and $w_2(z)$ is holomorphic in $\overline{\mathbb{C}}$ with a cut over $[x_a, 1]$. The system of equations relating the limits of the recurrence coefficients for the Angelesco system and the functions $\phi^{(0)}$, $\phi^{(2)}$, takes the following form (the upper index A means Angelesco system):

$$1 + a^A_{0,0}\phi^{(0)}_1(x) + a^A_{0,-1}\phi^{(0)}_1\phi^{(2)}_1(x) + a^A_{0,-2}\phi^{(0)}_1\phi^{(2)}_1\phi^{(0)}_1(x) = x\phi^{(0)}_1(x),$$

$$1 + a^A_{1,0}\phi^{(2)}_1(x) + a^A_{1,-1}\phi^{(2)}_1\phi^{(0)}_1(x) + a^A_{1,-2}\phi^{(2)}_1\phi^{(0)}_1\phi^{(2)}_1(x) = x\phi^{(2)}_1(x).$$

This implies the formulas

$$\phi^{(0)} = \frac{w + a_{0,-1}}{(x - a_{0,0})w - a_{0,-2}}, \qquad \phi^{(2)} = \frac{w + a_{1,-1}}{(x - a_{1,0})w - a_{1,-2}}.$$

Calculating the power series expansions of the functions $\phi^{(0)}$, $\phi^{(2)}$ at the points $\infty_0, \infty_1, \infty_2$ and using Lemmas 4.1 and 4.2, we obtain

Corollary 4.3. *For any Nikishin system $\mathcal{N}(\sigma_1, \sigma_2)$ associated with the system of intervals $\Delta_1 = [a, 0]$, $\Delta_2 = [x_a, 1]$ such that $\sigma'_j > 0$ a.e. on Δ_j, $j = 1, 2$, we have*

$$a^N_{0,0} = -\frac{1}{36R^2}[K - 12(a+1)R^2 + 10R^3], \quad a^N_{1,0} = -\frac{1}{18R^2}[K - 6(a+1)R^2 + 4R^3],$$

$$a^N_{0,-1} = -\frac{1}{36^2R^4}(-K+2R^3)(K-14R^3), \quad a^N_{1,-1} = -\frac{1}{36^2R^4}(-K+2R^3)(K-14R^3),$$

$$a^N_{0,-2} = -\frac{1}{36\cdot 81R^3}(-K+2R^3)^2, \quad a^N_{1,-2} = -\frac{1}{36^3R^6}(-K+2R^3)^2(K+2R^3).$$

This corollary follows from the following formulas for the coefficients of the power series expansions at infinity of the functions $\phi^{(0)}$ and $\phi^{(2)}$

$$c^{(0)}_0 = \frac{1}{81R}(-K+2R^3), \quad c^{(0)}_3 = -\frac{1}{36\cdot 81R^3}\,[K-12(a+1)R^2+10R^3](-K+2R^3),$$

$$c^{(1)}_1 = 1, \quad c^{(0)}_4 = -\frac{1}{3}\,(a+1) + \frac{2}{9}\,R,$$

$$c^{(0)}_2 = \frac{4}{9}\,R, \quad c^{(0)}_5 = \frac{1}{81R}\,(K+2R^3),$$

$$c^{(2)}_0 = -\frac{1}{16R^3}\,(-K+2R^3), \quad c^{(2)}_3 = \frac{1}{288R^5}\,[K - 6(a+1)R^2 + 4R^3](-K+2R^3),$$

$$c^{(2)}_1 = \frac{9}{4R}, \quad c^{(2)}_4 = 1,$$

$$c^{(2)}_2 = 81\,\frac{R}{K+2R^3}, \quad c^{(2)}_5 = -\frac{9}{4}\,\frac{[-K + 12(a+1)R^2 + 10R^3]}{R(K+2R^3)} + \frac{9}{4R}.$$

With this result, we obtain immediately the coefficients of the algebraic equations for the functions $\psi^{(1)}$ and $\psi^{(2)}$. They are

$$A_1 = 1, \quad B_1 = -\frac{1}{3}\,(a+1) + \frac{2}{3}\,R, \quad C_1 = \frac{4}{9}\,R, \quad D_1 = -\frac{4}{27}\,R(a+1-R),$$

$$E_1 = \frac{4}{9^3}\,(-K+2R^3),$$

$$A_2 = \frac{81R}{K+2R^3}, \quad B_2 = -\frac{27}{4}\frac{[-K+4(a+1)R^2+2R^3]}{R(K+2R^3)}, \quad C_2 = \frac{9^3}{4}\frac{1}{(K+2R^3)},$$

$$D_2 = \frac{243}{16}\frac{[K-4(a+1)R^2+2R^3]}{R^2(K+2R^3)}, \quad E_2 = -\left(\frac{9}{4}\right)^3\frac{(-K+2R^3)}{R^3(K+2R^3)}.$$

Example. Consider two intervals $\Delta_1 = [-1, 0]$ and $\Delta_2 = [0, 1]$. In this case $a = -1$. The reference Hessenberg operators are then defined for Angelesco and Nikishin systems as follows

$$A^0_{\mathrm{Ang}} = \begin{pmatrix} -\alpha & 1 & 0 & 0 & \dots & 0 \\ \alpha^2 & \alpha & 1 & 0 & \dots & 0 \\ -\alpha^3 & \alpha^2 & -\alpha & 1 & \dots & 0 \\ 0 & \alpha^3 & \alpha^2 & \alpha & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

$$A^0_{\mathrm{Nik}} = \begin{pmatrix} -\frac{5}{4}\alpha & 1 & 0 & 0 & \dots & 0 \\ \frac{7}{16}\alpha^2 & -\alpha & 1 & 0 & \dots & 0 \\ -\frac{1}{8}\alpha^3 & \frac{7}{16}\alpha^2 & -\frac{5}{4}\alpha & 1 & \dots & 0 \\ 0 & -\frac{1}{64}\alpha^3 & \frac{7}{16}\alpha^2 & -\alpha & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

where $\alpha = 2/(3\sqrt{3})$. The algebraic equations for the functions $\psi^{(1)}$ and $\psi^{(2)}$ take the form

$$\lambda^3 - \left(x + \frac{2}{3}\sqrt{3}\right)\lambda^2 + \left(\frac{4}{9}\sqrt{3}x + \frac{4}{9}\right)\lambda - \frac{8}{243}\sqrt{3} = 0$$

and

$$\lambda^3 - \left(\frac{27}{2}x - \frac{9}{4}\sqrt{3}\right)\lambda^2 + \left(\frac{81}{8}\sqrt{3}x + \frac{81}{16}\right)\lambda + \frac{81}{64}\sqrt{3} = 0.$$

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